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Boundary-layer development at a two-dimensional rear stagnation point

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This paper examines the nature of the development of two-dimensional laminar flow of an incompressible fluid at the rear stagnation point on a cylinder which is started impulsively from rest. Proudman & Johnson (1962) first examined this type of flow, and obtained a similarity solution of the inviscid form of the equations of motion. This solution describes the nature of the flow at large distances from the surface, for large times after the start of the motion. Here, the flow at the rear stagnation point is examined in greater detail. The solution found by Proudman & Johnson constitutes the leading term in an asymptotic expansion, valid for large times. Further terms in this expansion are now calculated, and the method of matched asymptotic expansions is used to obtain an inner solution describing the flow near the surface. A numerical integration of the full initial-value problem gives good agreement with the analytical solution.

1. Introduction

Consider the development of the two-dimensional laminar flow of an incompressible fluid of small viscosity past a cylinder which is started impulsively from rest and then maintained at a constant speed. At the initial instant of starting, the classical potential flow prevails throughout the entire flow field, save for a layer of intense vorticity concentrated on the surface of the cylinder. For small times after the start of the motion the development of the boundary layer on the surface is described by the series solution of Blasius (1908) and Goldstein & Rosenhead (1936). (Wundt (1955) corrected errors in Goldstein & Rosenhead.)

The existence of an adverse pressure gradient over the rear of the cylinder leads to a thickening of the boundary layer, accompanied by a decrease in the skin friction, until eventually a time is reached when the skin friction vanishes at some point on the surface, and flow reversal begins. The precise location of this point depends on the shape of the surface, but for a circular cylinder the flow reversal begins at the rear stagnation point. The onset of flow reversal leads to the establishment of regions of closed streamlines at the rear of the cylinder, the size of which increases rapidly with time. The flow at the front of the cylinder rapidly approaches a steady state, but it is well known that no solutions of the boundary-layer equations representing steady flow near the rear stagnation point can be found.

An important contribution to the understanding of the flow development at the rear stagnation point has been made by Proudman & Johnson (1962). They argued that, since the boundary-layer thickness grows very rapidly under the action of the convection field once flow reversal has occurred, the length scale normal to the boundary becomes much larger than the distance over which viscous forces are important. They therefore conjectured that the viscous term in the governing equation is important only near the boundary and that most of the asymptotic flow for large times is governed by the inviscid equation. A similarity solution of the inviscid equation was found, and showed that the boundary-layer thickness increases exponentially with time. This solution also showed that the flow near the boundary ultimately becomes steady flow towards a stagnation point. The skin friction at the rear stagnation point therefore tends to a finite negative value, equal in magnitude to that at the front stagnation point.

We now examine the rear stagnation point in greater detail. The similarity solution found by Proudman & Johnson constitutes the leading term in an asymptotic expansion describing the flow at large distances from the boundary. Further terms in this expansion are now determined. In conjunction with this, an inner expansion which describes the flow near the boundary for large times after the start of the motion is developed. The two expansions are matched in the usual manner. It will be seen that a number of arbitrary constants arise in the solution, not all of which can be determined by matching the two expansions. It is not surprising that a certain amount of indeterminacy appears in the solution, since the expansions proceed 'backwards' in time, whereas the flow pattern at any instant is of course determined by details of the flow at earlier times.

In order to determine the approximate value of these constants, a numerical solution of the full initial-value problem is undertaken. Suitable choice of the constants enables a fairly close match to be made between the numerical and analytical solutions.

2. Equations of motion and boundary conditions

Let x' and y' be co-ordinates measured, respectively, along and normal to the surface of the cylinder, where x' is measured away from the rear stagnation point. We define non-dimensional co-ordinates x, y and t by

$$x = x'/a, \quad y = y'(2U_0/\nu a)^{\frac{1}{2}}, \quad t = 2U_0 t'/a, \tag{2.1}$$

where U_0 is the speed of the stream at infinity, a is the radius of the cylinder, ν is the kinematic viscosity of the fluid and t' the time. The flow in the neighbourhood of the rear stagnation point is idealized by considering the surface to be an infinite plane wall. The potential flow corresponding to an impulsive start is then described by the stream function

$$\psi = -(2\nu a U_0)^{\frac{1}{2}} xy, \tag{2.2}$$

giving
$$u = \partial\psi/\partial y' = -2U_0 x, \quad v = -\partial\psi/\partial x' = (2\nu U_0/a)^{\frac{1}{2}} y, \tag{2.3}$$

where u and v are the velocity components along and normal to the surface. Since the flow field is assumed to remain unchanged at sufficiently large distances from the boundary at any finite time, the above potential flow will be maintained as an outer boundary condition for all values of t.

Provided that the surface is an infinite plane wall, we can obtain an exact solution of the Navier–Stokes equations by writing the stream function in the form

$$\psi = -(2\nu a U_0)^{\frac{1}{2}} x F(y, t), \tag{2.4}$$

where the function $F(y,t)$ must satisfy the boundary and initial conditions

$$\left.\begin{aligned} F = F_y \quad \text{on} \quad y = 0 \quad \text{for} \quad t \neq 0, \\ F_y \to 1 \quad \text{as} \quad y \to \infty, \\ F_y = 1 \quad \text{at} \quad t = 0 \quad \text{for} \quad y \neq 0. \end{aligned}\right\} \tag{2.5}$$

The equations of motion then yield the following equation for F:

$$F_{yyy} - F F_{yy} - 1 + F_y^2 = F_{yt}. \tag{2.6}$$

Our objective is to obtain an asymptotic solution describing the flow at large times after the start, so the details of the initial flow need not be considered here. Proudman & Johnson found a similarity solution of the inviscid form of (2.6), in the form

$$F(y,t) = e^t f(y e^{-t}). \tag{2.7}$$

This solution describes the exponential growth of the boundary-layer thickness for large values of t. Hence the relevant outer variable is given by $\eta = y e^{-t}$, and we write the stream function describing the flow in the outer region as

$$\psi_0 = -(2\nu a U_0)^{\frac{1}{2}} x G(\eta, t) \quad (\eta = y e^{-t}). \tag{2.8}$$

Substitution into the Navier–Stokes equations gives the following equation for G:

$$e^{-3t} G_{\eta\eta\eta} - e^{-2t} G G_{\eta\eta} - 1 + e^{-2t} G_\eta^2 = e^{-t}(G_{\eta t} - G_\eta - \eta G_{\eta\eta}), \tag{2.9}$$

where $G(\eta,t)$ is required to satisfy the outer boundary condition

$$G_\eta(\eta, t) \to e^t \quad \text{as} \quad \eta \to \infty, \quad \text{i.e.} \quad \text{as} \quad y \to \infty \quad \text{for fixed } t. \tag{2.10}$$

Near the boundary, where the viscous forces are important, we seek a solution satisfying the inner boundary conditions of zero normal and tangential velocity, so we write the stream function in the form

$$\psi_i = -(2\nu a U_0)^{\frac{1}{2}} x g(y, t), \tag{2.11}$$

where $g(y,t)$ satisfies the inner boundary conditions

$$g = g_y = 0 \quad \text{on} \quad y = 0. \tag{2.12}$$

The inner equation is then given by

$$g_{yyy} - g g_{yy} - 1 + g_y^2 - g_{yt} = 0. \tag{2.13}$$

3. The analytical solution

As was previously indicated, we expect the Proudman & Johnson solution to form the leading term of our outer expansion. Hence, for our first outer solution, we write

$$G(\eta, t) = e^t F_0(\eta).$$

Substitution in (2.9) yields the equation obtained by Proudman & Johnson:

$$(F_0-\eta)F_0''+1-F_0'^2 = 0, \tag{3.1}$$

with solution

$$F_0 = \eta-(2/c)(1-e^{-c\eta}), \tag{3.2}$$

where c is a constant of integration whose value will be determined in the next section. We expand this outer solution and rewrite it in inner variables to obtain

$$G(\eta,t) \sim -y+cy^2e^{-t}-\tfrac{1}{3}c^2y^3e^{-2t}+O(e^{-3t}). \tag{3.3}$$

The leading term of this expansion is independent of t, indicating that our first inner solution is

$$g(y,t) = f_0(y). \tag{3.4}$$

Substitution in (2.13) then yields

$$\left.\begin{aligned} &f_0''''-f_0f_0''-1+f_0'^2 = 0,\\ &f_0(0) = f_0'(0) = 0, \quad f_0(y)\to -y \quad \text{as} \quad y\to\infty.\end{aligned}\right\} \tag{3.5}$$

This is effectively the forward-stagnation-point boundary-layer problem (Hiemenz 1911) with a change in sign in f_0. It is a numerical problem, and, as is well known, the correct behaviour as $y\to\infty$ is ensured by choosing $f_0''(0)$ suitably. With $f_0''(0) = -1{\cdot}2326\ldots$, the asymptotic form of f_0 is

$$f_0 \sim -y+\delta+\exp, \tag{3.6}$$

where $\delta = 0{\cdot}6479\ldots$, and 'exp' denotes a term exponentially small as $y\to\infty$.

Thus the leading term matches with the outer expansion, and the displacement constant δ indicates that the outer expansion should proceed as

$$G(\eta,t) = e^tF_0(\eta)+F_1(\eta). \tag{3.7}$$

Substitution in (2.9) then gives

$$\left.\begin{aligned} (\eta-F_0)F_1''+(2F_0'+1)F_1'-F_0''F_1 &= 0,\\ F_1'(\infty) &= 0.\end{aligned}\right\} \tag{3.8}$$

The solution is

$$F_1 = A_1(\tfrac{1}{2}-e^{-c\eta})+A_1'\{e^{-\frac{1}{2}c\eta}(1-e^{-c\eta})^{\frac{1}{2}}-(1-2e^{-c\eta})\sin^{-1}(e^{-\frac{1}{2}c\eta})\}. \tag{3.9}$$

We expand for small η and rewrite in inner variables to obtain

$$\begin{aligned} G(\eta,t) &= e^tF_0+F_1\\ &\sim -y+(\tfrac{1}{2}\pi A_1'-\tfrac{1}{2}A_1)+\{y^2+(A_1-A_1'\pi)y\}ce^{-t}\\ &\quad+\tfrac{4}{3}A_1'y^{\frac{3}{2}}c^{\frac{3}{2}}e^{-\frac{3}{2}t}+\{-\tfrac{1}{3}y^3+\tfrac{1}{2}(\pi A_1'-A_1)y^2\}c^2e^{-2t}+O(e^{-\frac{5}{2}t}). \end{aligned} \tag{3.10}$$

To match with our inner solution we must take

$$\tfrac{1}{2}\pi A_1'-\tfrac{1}{2}A_1 = \delta. \tag{3.11}$$

Note that A_1 and A_1' are not completely determined. It is convenient to regard A_1' as being unknown at this stage, and regard A_1 as being known in terms of A_1'. Thus we write $A_1 = -2\delta+\pi A_1'$.

It is relevant at this stage to comment on a remark in the original Proudman & Johnson paper. They refer (in an immediate context which need not directly concern us here) to the "second term of the asymptotic expansion for large t", and say that "one solution involves fractional powers of $y e^{-t}$ near $y = 0$ and seems unlikely to be relevant". This is equivalent to the assertion that $A_1' = 0$. We shall show that this is not the case (although this does not affect the validity of the qualitative argument with which they are directly concerned). Certainly one cannot rule out that A_1' terms on the grounds of singular derivatives at $y = 0$, because the terms in question are part of an outer solution, not an inner one. In fact we shall see that the A_1' term gives rise to a homogeneous inner equation, whose solution is determined in terms of A_1'.

It is clear that our inner solution must proceed as

$$g(y,t) = f_0(y) + c e^{-t} f_1(y), \tag{3.12}$$

where, from (2.13),

$$\left.\begin{aligned} f_1''' - f_0 f_1'' + (2f_0' + 1) f_1' - f_0'' f_1 &= 0, \\ f_1(0) = f_1'(0) &= 0. \end{aligned}\right\} \tag{3.13}$$

The asymptotic form of this equation is given by

$$f_1 \sim \alpha_1 (y^2 - 2\delta y) + \alpha_1' + \exp, \tag{3.14}$$

for some constants α_1 and α_1'. Since we have already chosen $A_1 - \pi A_1' = -2\delta$, we can match this solution with the outer one by taking $\alpha_1 = 1$. This is achieved by choosing $f_1''(0)$ suitably in the numerical integration of the inner problem. We find that $f_1''(0) = 1{\cdot}6337\ldots$; α_1' is then determined, and is equal to $1{\cdot}7060\ldots$.

It is convenient at this stage to match the $e^{-\frac{3}{2}t}$ term in the outer solution, so that our complete second inner solution is of the form

$$g(y,t) = f_0(y) + c e^{-t} f_1(y) + c^{\frac{3}{2}} e^{-\frac{3}{2}t} f_{\frac{3}{2}}(y), \tag{3.15}$$

where, from (2.13),

$$f_{\frac{3}{2}}''' - f_0 f_{\frac{3}{2}}'' + (2f_0' + \tfrac{3}{2}) f_{\frac{3}{2}}' - f_0'' f_{\frac{3}{2}} = 0, \tag{3.16}$$

$$f_{\frac{3}{2}}(0) = f_{\frac{3}{2}}'(0) = 0. \tag{3.17}$$

The asymptotic form is

$$f_{\frac{3}{2}} \sim \alpha_{\frac{3}{2}} [y^{\frac{3}{2}} - \tfrac{3}{2}\delta y^{\frac{1}{2}} + \tfrac{3}{4}\delta^2 y^{-\frac{1}{2}} + O(y^{-\frac{3}{2}})] + \alpha_{\frac{3}{2}}' + \exp, \tag{3.18}$$

and to match with the outer solution we must take

$$\alpha_{\frac{3}{2}} = \tfrac{4}{3} A_1'. \tag{3.19}$$

Hence $\alpha_{\frac{3}{2}}$ is known only in terms A_1', which itself is not known. Such indeterminacy is not uncommon in problems of matched asymptotic expansions. The value of the constant depends on the precise nature of the earlier time development, which is known only from a full study of the viscous initial-value problem. Strictly speaking, this is the second undetermined constant to arise, the first being the constant c of the original Proudman & Johnson analysis. In fact the constant c represents an uncertainty in the precise location of the time origin. This can readily be seen by noting that, in the outer problem, c always occurs multiplying the variable η; thus $c\eta = cy e^{-t}$. Clearly a change from t to $t + t^*$ can be incorporated in c. Later, an attempt is made to determine A_1' from the numerical solution of the full viscous problem.

It is clear that the term α_1' in f_1 will generate the third outer solution, which must proceed as

$$G(\eta, t) = e^t F_0(\eta) + F_1(\eta) + e^{-t} F_2(\eta),$$

where, from (2.9),

$$\left.\begin{aligned}(\eta - F_0) F_2'' + 2(F_0' + 1) F_2' - F_0'' F_2 &= -F_0''' + F_1 F_1'' - F_1'^2,\\ F_2'(\infty) &= 0.\end{aligned}\right\} \tag{3.20}$$

The solution, after some algebra, is

$$\begin{aligned}F_2 = c(\tfrac{1}{4}A_1'^2 - 1) - cA_1A_1' e^{-c\eta} \sin^{-1}(e^{-\frac{1}{2}c\eta}) + cA_1'^2 e^{-c\eta} [\sin^{-1}(e^{-\frac{1}{2}c\eta})]^2 \\ + cA_2(1 - e^{-c\eta}) + cA_2' \{(1 - e^{-c\eta}) \log (1 - e^{c\eta}) + 1\},\end{aligned} \tag{3.21}$$

for some constants A_2 and A_2'.

If we expand for small η, rewrite in inner variables and replace A_1 by $\pi A_1' - 2\delta$, as previously determined, we obtain

$$\begin{aligned}&G(\eta, t) - e^t F_0(\eta) + F_1(\eta) + e^{-t} F_2(\eta)\\ &\quad \sim -y + \delta + \{y^2 - 2\delta y + (A_2' + \delta^2 - 1)\} c e^{-t}\\ &\quad + A_1'(\tfrac{4}{3} y^{\frac{3}{2}} - 2\delta y^{\frac{1}{2}}) c^{\frac{3}{2}} e^{-\frac{3}{2}t} - A_2' y c^2 t e^{-2t}\\ &\quad + \{-\tfrac{1}{3}y^3 + \delta y^2 + A_2' y \log y + [A_1'^2(1 + \tfrac{1}{4}\pi^2) - \pi\delta A_1' + A_2 + A_2' \log c] y\} c^2 e^{-2t}\\ &\quad + O(e^{-\frac{5}{2}t}).\end{aligned} \tag{3.22}$$

Note first that the extra term $-2\delta y^{\frac{1}{2}} A_1'$ in the $c^{\frac{3}{2}} e^{-\frac{3}{2}t}$ bracket already matches with a corresponding inner term, by the choice of $\alpha_{\frac{3}{2}}$ made already.

To match the $\alpha_1' c e^{-t}$ term in the inner expansion, we must now take

$$\alpha_1' = A_2' + \delta^2 - 1;$$

that is,

$$A_2' = \alpha_1' + 1 - \delta^2 = 2{\cdot}2862\ldots. \tag{3.23}$$

Clearly our third inner solution must be of the form

$$g(y, t) = f_0(y) + c e^{-t} f_1(y) + c^{\frac{3}{2}} e^{-\frac{3}{2}t} f_{\frac{3}{2}}(y) + c^2 t e^{-2t} f_{21}(y) + c^2 e^{-2t} f_{20}(y). \tag{3.24}$$

Note that, if $\epsilon = e^{-t}$, then $t e^{-2t} = -\epsilon^2 \log \epsilon$, and so this is the first stage at which logarithmic terms enter the expansion. We find that

$$\begin{aligned}f_{21}''' - f_0 f_{21}'' + 2(f_0' + 1) f_{21}' - f_0'' f_{21} &= 0,\\ f_{21}(0) = f_{21}'(0) &= 0\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}f_{20}''' - f_0 f_{20}'' + 2(f_0' + 1) f_{20}' - f_0'' f_{20} &= f_{21}' + f_1 f_1'' - f_1'^2,\\ f_{20}(0) = f_{20}'(0) &= 0.\end{aligned} \tag{3.26}$$

Asymptotic analysis gives

$$f_{21} \sim \alpha_{21} y + \alpha_{21}' + \exp, \tag{3.27}$$

$$f_{20} \sim -\tfrac{1}{3}y^3 + \delta y^2 + \alpha_{20} y + [\alpha_{21} + 2(1 + \alpha_1' - \delta^2)] y \log y + \alpha_{20}' + O(y^{-1}) + \exp, \tag{3.28}$$

for some constants α_{21}, α_{21}', α_{20} and α_{20}'.

We now do the matching. For the $t e^{-2t}$ term we need

$$\alpha_{21} = -A_2' = -2{\cdot}2862\ldots; \tag{3.29}$$

this is done by choosing $f''_{21}(0) = -2{\cdot}4954\ldots$ in the numerical integration. This in turn determines α'_{21} as $1{\cdot}1853\ldots$.

Now consider the term of order e^{-2t}. We match the term $y \log y$ by taking

$$\alpha_{21} + 2(1 + \alpha'_1 - \delta^2) = A'_2.$$

This condition is already satisfied since we have chosen

$$\alpha_{21} = -A'_2 = -(1 + \alpha'_1 - \delta^2).$$

Finally the term of order $y e^{-2t}$ is matched by taking

$$\alpha_{20} = A'^2_1(1 + \tfrac{1}{4}\pi^2) - \pi\delta A'_1 + A'_2 \log c + A_2. \tag{3.30}$$

We see that α_{20} is known only in terms of A'_1 and A_2. Thus A_2 is the next undetermined constant to arise. The outer expansion will proceed further, with terms of order $e^{-\frac{5}{2}t}$, te^{-2t}, e^{-2t}, etc. However, there seems little point in continuing the analysis further, partly because of the algebraic complexity and partly because the solutions would be of doubtful use because of many further undetermined constants which will arise, and be extremely difficult to evaluate.

Finally, we ought to consider the effect of adding linearized eigensolutions of the inner problem to our solution. Thus, if we consider

$$g(y,t) = f_0(y) + \epsilon f^*(y,t),$$

where $\quad f^*(0,t) = f^*_y(0,t) = 0 \quad \text{and} \quad f^*_y(y,t) \to 0$

exponentially as $y \to \infty$, we obtain, on linearization,

$$f^{*'''} - f_0 f^{*''} + 2f'_0 f^{*'} - f''_0 f^* = f^{*'}_t.$$

This is the problem considered by Kelly (1962) (with $-f_0$ for f_0). The first linearized eigenfunction is of the form $H(y)e^{-\lambda t}$, where $\lambda > 0$ is an eigenvalue, whose lowest possible value is $3{\cdot}063\ldots$. This term is thus of higher order than those we have considered, and may be legitimately ignored at this stage. Note that, since these eigenfunctions are exponentially small as $y \to \infty$, they do not affect the outer solution, nor are they affected by it.

4. Numerical solution

A numerical solution of (2.6) subject to the boundary conditions (2.5) was undertaken in order to estimate the values of the constants c, A'_1 and A_2. A Crank–Nicolson fully implicit finite-difference technique was used. This method has the advantage of being unconditionally stable, imposing no restrictions on the mesh intervals used. Since the equation is nonlinear, we must use an iterative technique. The solution at any time step was deemed to have converged when two successive evaluations of the skin friction differed by less than a small tolerance ϵ, which was taken to be 10^{-5}. Each iteration involves the inversion of a tri-diagonal matrix. A velocity profile must be assumed at the beginning of each time step, to start off the iteration. This is usually taken as the profile at the previous time step, except at the start of the integration, when a straight-line velocity profile is assumed.

The mesh intervals used were $\Delta t = \Delta y = 0{\cdot}05$, so that 110 time steps were taken in integrating the equations from $t = 0$ to $t = 5{\cdot}5$. To allow for the rapid growth of the boundary-layer thickness at large times, the number of mesh points across the layer was increased at certain preassigned values of t. The value of y_∞, at which the outer boundary condition is enforced, was eventually increased to 500, that is, 10 000 steps.

As was noted by Proudman & Johnson, it seems likely that the value of the constant c is determined by the early development of the flow. It was therefore thought important to obtain a fairly accurate solution at the start of the motion. The relevant boundary-layer co-ordinate for small times is

$$y_1 = \tfrac{1}{2}y'(\nu t')^{-\frac{1}{2}}, \tag{4.1}$$

and the initial profile is given (with non-dimensional velocity u) by

$$u = \operatorname{erf} y_1 \tag{4.2}$$

(Blasius 1908). The boundary-layer finite-difference equations were therefore initially formulated using y_1 as the co-ordinate normal to the surface; the integration was performed up to $t = 0{\cdot}25$, and after this point the co-ordinate $y_2 = y'(2U_0/\nu a)^{\frac{1}{2}}$ was introduced. The integration was then continued up to $t = 5{\cdot}5$ without any further change of variable. The value of $t = 0{\cdot}25$ was chosen since $y_1 = y_2$ at $t = 0{\cdot}25$ for all values of y'; that is, the two finite-difference grids coincide at this value of t, so that the velocity profile at $t = 0{\cdot}25$ may be used directly to calculate the profile at the next time step, without the use of any interpolation procedure.

The value of the constant c was estimated using the method adopted by Proudman & Johnson. The leading term of the outer expansion gives

$$\log(1-u) = -c\eta + \log 2, \tag{4.3}$$

so that, for any fixed large value of t, a graph of $\log(1-u)$ against η should yield a straight line of gradient $-c$, except of course for small values of y, when the expansion is invalid. Two values of t were tried, $t = 4{\cdot}5$ and $t = 5{\cdot}0$. The method of least squares was used to fit a straight line to the points, using data from the range $15 \leqslant y \leqslant 90$ for $t = 4{\cdot}5$ and $15 \leqslant y \leqslant 145$ for $t = 5{\cdot}0$. The value of c that emerged was $3{\cdot}51$ in each case, with a standard error of about $0{\cdot}003$. This differs significantly from Proudman & Johnson's estimate, $c = 3{\cdot}8$. However, since the value of c is determined by the initial flow, it seems likely that the present estimate is the more accurate, for the following reasons.

The numerical integration was started off by using the co-ordinate

$$y_1 = \tfrac{1}{2}y'(\nu t)^{-\frac{1}{2}},$$

the initial profile being given by $u = \operatorname{erf} y$. Since $\operatorname{erf} z = 0{\cdot}99997$ when $z = 3$, this corresponds to taking about 60 points across the layer, since Δy was chosen as $0{\cdot}05$. However, Proudman & Johnson used the co-ordinate y_2 throughout, and started off the integration from $t = 0{\cdot}0025$, the initial data being found from the series solution for small times. Hence the initial velocity profile in this case is given

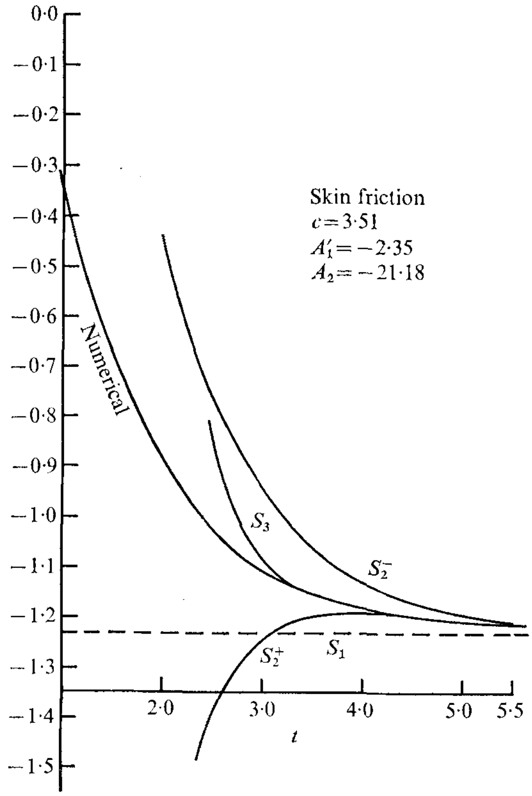

FIGURE 1. Two-dimensional skin friction. $c = 3{\cdot}51$, $A_1' = -2{\cdot}35$, $A_2 = -21{\cdot}18$.

approximately by $u = \operatorname{erf}(y_2/2t^{\frac{1}{2}}) = \operatorname{erf}(10y_2)$ at $t = 0{\cdot}0025$. Thus the whole of the variation in u is initially confined to the region $0 \leqslant y_2 \leqslant 0{\cdot}3$, which corresponds to only a few mesh lengths in the y direction and presumably gives less accurate starting values than in the present case. Presumably, any inaccuracy in the initial profile would mean that a large time interval would have to elapse before the emergence of the asymptotic similarity solution. This in turn would be reflected in a change in the value of c, as was indicated in the previous section. For the purposes of estimating the values of the constants A_1' and A_2 we shall therefore assume that $c = 3{\cdot}51$ is a fairly accurate estimate.

5. Comparison with numerical results

As has been indicated, the constants A_1' and A_1 must be determined from the numerical solution. This is not a trivial task. The best way seems to be by choosing A_1' and A_2 to give the best agreement between analytical and numerical curves of skin friction against time. The curves of displacement thickness against time then give some sort of cross-check. Approximately, this gives $f_{\frac{3}{2}}''(0) = -4{\cdot}1$, and hence $\alpha_{\frac{3}{2}}' = -0{\cdot}86$. This then gives $A_1' = -2{\cdot}3$ and $A_1 = -7{\cdot}4$. Also $f_{20}''(0) = 13$, $\alpha_{20} = 5{\cdot}6$, $A_2 = -21$ and $\alpha_{20}' = 1{\cdot}1$.

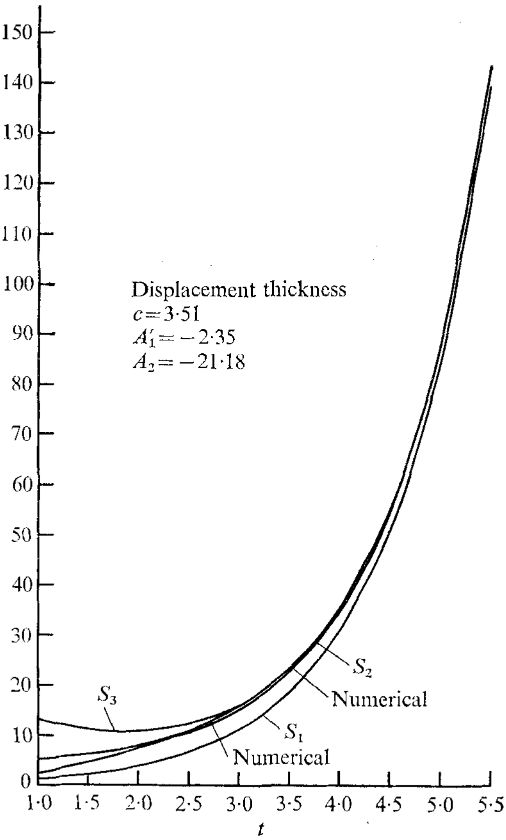

FIGURE 2. Two-dimensional displacement thickness.
$c = 3{\cdot}51$, $A_1' = -2{\cdot}35$, $A_2 = -21{\cdot}18$.

Clearly these figures are only very approximate. Their sole merit is that they give the best fit with the expansion as far as it has been taken, and we have no means of estimating the effect of further undetermined constants in higher order terms. However, with these values, good agreement is obtained with numerical results back to about $t = 3{\cdot}25$. (It may be interesting to point out that the three-term Goldstein & Rosenhead expansion about $t = 0$ gives reasonable agreement with the skin friction curve out to about $t = 0{\cdot}8$.)

We have

$$g''(0,t) \sim f_0''(0) + c\,e^{-t} f_1''(0) + c^{\frac{3}{2}} e^{-\frac{3}{2}t} f_{\frac{3}{2}}''(0) + c^2 t\,e^{-2t} f_{21}''(0) + c^2 e^{-2t} f_{20}''(0).$$

The curves in figure 1 are as follows:

$$S_1 = f_0''(0), \quad S_2^- = f_0''(0) + c\,e^{-t} f_1''(0), \quad S_2^+ = S_2^- + c^{\frac{3}{2}} e^{-\frac{3}{2}t} f_{\frac{3}{2}}''(0),$$
$$S_3 = S_2^+ + c^2 t\,e^{-2t} f_{21}''(0) + c^2 e^{-2t} f_{20}''(0).$$

Also, the displacement thickness of the whole flow, say Δ, is given by

$$\Delta = \int_0^\infty (1-F')\,dy \sim \frac{2}{c} e^t - \frac{A_1}{2} - c\left\{\frac{A_1'^2}{4} - 1 + A_2 + A_2'\right\} e^{-t}.$$

The curves in figure 2 are as follows:

$$S_1 = (2/c)\,e^t, \quad S_2 = S_1 - \tfrac{1}{2}A_1, \quad S_3 = S_2 - c\{\tfrac{1}{4}A_1'^2 - 1 + A_2 + A_2'\}e^{-t}.$$

6. Conclusions

The foregoing analysis constitutes an exact solution of the Navier–Stokes equations in the idealized case of an infinite plane boundary. However, it should be remembered that for a cylinder, having non-zero curvature, the solution is a boundary-layer approximation only, and is valid only at the rear stagnation point $x = 0$. In order to examine the flow for non-zero values of x, we must revert to the full boundary-layer equations. This problem is now being studied for the case of a circular cylinder.

In this paper we have examined only the case of two-dimensional stagnation-point flow. However, work has also been carried out for the case of flow at an axisymmetric stagnation point, and indeed, for the general three-dimensional case. It is hoped to make of all these studies the subject of later publications.

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